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Boundary-value problems in the kinetic theory of gases. Part 2. Thermal creep

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The effect of a temperature gradient in a gas inclined at an angle to a boundary wall has been investigated. For an infinite half-space of gas it is found that, in addition to the conventional temperature slip problem, the component of the temperature gradient parallel to the wall induces a net mass flow known as thermal creep. We show that the temperature slip and thermal creep effects can be decoupled and treated quite separately.

Expressions are obtained for the creep velocity and heat flux, both far from and at the boundary; it is noted that thermal creep tends to reduce the effective thermal conductivity of the medium.

1. Introduction

In a recent publication (Williams 1969, referred to as I) we have presented a general technique for solving boundary-value problems in gas-kinetic theory. The method was illustrated by application to the problem of slip flow in a half-space, i.e. the Kramers problem. We now wish to apply our technique to a problem involving a combination of temperature and velocity gradients and will therefore consider a generalization of the usual temperature jump problem (Welander 1954).

The problem under consideration is that of a half-space ($x > 0$) filled with a rarefied gas and bounded in the plane $x = 0$ by a wall. A temperature gradient is maintained in the gas; however, unlike the usual temperature jump problem, the gradient is not normal to the wall but inclined to it at some angle. Thus the gradient is defined by its two components dT/dx normal to the wall, and dT/dz parallel to the wall.

When linearized about a local Maxwellian, the Boltzmann equation assumes a form for which the solution may be shown to decouple; one part describing the normal temperature jump condition, the other describing a phenomenon known as *thermal creep*. The latter effect is explained in detail by Kennard (1938) who shows that at any differentially heated boundary there exists a tendency for a gas to move along the surface from colder to hotter regions. In enclosed vessels this effect is known as thermal transpiration and has been studied recently in some detail by Sone & Yamamoto (1968), Loyalka (1969) and Williams (1970). A classic example of thermal transpiration in the Knudsen limit is the porous plug experiment (Kennard 1938). For enclosed vessels a 'back pressure' is generated which induces Poiseuille flow as well as thermal creep; since these effects act in opposite directions an interesting competition takes place which, under certain conditions, can be arranged to give zero net flow of gas. However in the half-space problem considered above, we shall assume that, asymptotically, the pressure is constant and this will lead to a pure thermal creep velocity.

Our results for the thermal creep velocity are quite new, whilst those for the temperature jump lead to equations which have been solved by Loyalka & Ferziger (1968) and by Loyalka (1968). As two special cases, we employ the B.G.K. model of scattering, with constant collision frequency and constant collision cross-section, to calculate the thermal creep velocity and heat flux. Also, comparison is made with the simple kinetic theory results of earlier workers.

2. Basic theory

The Boltzmann equation may be written in the form

$$\mathbf{v}.\nabla f(\mathbf{v},\mathbf{r}) = n\hat{J}(f,f_1) \tag{1}$$

as explained in I.

An implicit assumption in our work on rarefied gases is that any perturbations from the equilibrium Maxwellian distribution must be small. Thus we look for solutions to (1) in the form

$$f(\mathbf{v},\mathbf{r}) = f_0(\mathbf{v},\mathbf{r})\{1+h(\mathbf{v},\mathbf{r})\}, \tag{2}$$

where $f_0(\mathbf{v},\mathbf{r})$ is the *local* Maxwellian and $h(\mathbf{v},\mathbf{r})$ is, in an average sense, small compared with unity.

The co-ordinate system is defined so that x is the distance measured normally from the wall and z is the distance measured along it; we can therefore define our local Maxwellian in the following manner:

$$f_0(\mathbf{v},\mathbf{r}) = n(x,z)\left(\frac{m}{2\pi kT(x,z)}\right)^{\frac{3}{2}}\exp\left\{-\frac{mv^2}{2kT(x,z)}\right\}, \tag{3}$$

where

$$n(x,z) = n_0\left(1+\frac{n_x}{n_0}x+\frac{n_z}{n_0}z\right) \tag{4}$$

and

$$T(x,z) = T_0\left(1+\frac{T_x}{T_0}x+\frac{T_z}{T_0}z\right), \tag{5}$$

together with the relation $nkT = p =$ constant; this gives

$$\frac{n_x}{n_0} = -\frac{T_x}{T_0} \quad\text{and}\quad \frac{n_z}{n_0} = -\frac{T_z}{T_0}, \tag{6}$$

where n_0 and T_0 are convenient reference values of density and temperature ($T_0+T_z z$ is the wall temperature).

The subscript x or z indicates a gradient with respect to that variable.

Linearizing $f_0(\mathbf{v},\mathbf{r})$ for small values of n_x/n_0, n_z/n_0, T_x/T_0 and T_z/T_0, and using (6), leads to the following expression for $f_0(\mathbf{v},\mathbf{r})$:

$$f_0(\mathbf{v},\mathbf{r}) = n_0\left(\frac{m}{2\pi kT_0}\right)^{\frac{3}{2}}e^{-mv^2/2kT_0}\left\{1+\left(\frac{mv^2}{2kT_0}-\frac{5}{2}\right)(K_x x+K_z z)\right\}, \tag{7}$$

where $K_x = T_x/T_0$ and $K_z = T_z/T_0$ are constants.

Inserting (2) and (7) into the Boltzmann equation (1) and neglecting second-order terms, lead to the following equation for h:

$$K_x c_x(c^2-\tfrac{5}{2})+K_z c_z(c^2-\tfrac{5}{2})+c_x\frac{\partial h(\mathbf{c},x)}{\partial x}+V(c)\,h(\mathbf{c},x) = \int d\mathbf{c}'K(\mathbf{c},\mathbf{c}')\,e^{-c'^2}\,h(\mathbf{c}',x). \quad (8)$$

In (8), the collision frequency is $V(c)$ and the scattering kernel is $K(\mathbf{c},\mathbf{c}')$, also we have so normalized the velocity that $c = v(m/2kT_0)^{\frac{1}{2}}$.

Now we change to polar co-ordinates as described by figure 1 of I. Defining $g(c,\mu,x)$ and $p(c,\mu,x)$ as follows:

$$g(c,\mu,x) = \frac{1}{\pi}\int_0^{2\pi} d\chi\cos\chi\, h(\mathbf{c},x), \quad (9)$$

$$p(c,\mu,x) = \frac{1}{2\pi}\int_0^{2\pi} d\chi\, h(\mathbf{c},x), \quad (10)$$

we see that (8) decouples for the two quantities g and p.

It is found, in fact, that

$$K_x c(c^2-\tfrac{5}{2})\mu+c\mu\frac{\partial p(c,\mu,x)}{\partial x}+V(c)\,p(c,\mu,x) = \sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\mu)\int_0^{\infty}dc'c'^2e^{-c'^2}K_l(c,c')\int_{-1}^{1}d\mu'\,P_l(\mu')\,p(c',\mu',x) \quad (11)$$

and

$$K_z c(c^2-\tfrac{5}{2})\,(1-\mu^2)^{\frac{1}{2}}+c\mu\frac{\partial g(c,\mu,x)}{\partial x}+V(c)\,g(c,\mu,x) = \sum_{l=1}^{\infty}\frac{2l+1}{2l(l+1)}P_l^{(1)}(\mu)\int_0^{\infty}dc'c'^2e^{-c'^2}K_l(c,c')\int_{-1}^{1}d\mu'\,P_l^{(1)}(\mu')\,g(c',\mu',x), \quad (12)$$

where the expansion of $K(\mathbf{c},\mathbf{c}')$ in associated spherical harmonics has been employed as explained in I.

Associated with (8) is the following boundary condition at $x = 0$:

$$h(c,\mu,\chi,0) = \frac{2}{\pi}\beta\int_0^1 d\mu'\mu'\int_0^{2\pi}d\chi'\int_0^{\infty}dc'c'^3e^{-c'^2}h(c',-\mu',\chi',0)+(1-\beta)\,h(c,-\mu,\chi,0) \quad (13)$$

for $\mu > 0$ and $0 \leqslant \chi \leqslant 2\pi$. In (13) we assume a mixture of specular and diffuse reflexion of molecules from the surface; β is the proportion of particles undergoing perfect accommodation. In terms of g and p, (13) becomes

$$g(c,\mu,0) = (1-\beta)\,g(c,-\mu,0) \quad (14)$$

and $$p(c,\mu,0) = 4\beta\int_0^1 d\mu'\mu'\int_0^{\infty}dc'c'^3e^{-c'^2}p(c',-\mu',0)+(1-\beta)\,p(c,-\mu,0) \quad (15)$$

for $\mu > 0$.

We shall return to these equations after defining some quantities of interest.

3. Macroscopic variables

The component P_{xz} of the pressure tensor is defined as

$$P_{xz} = m\left(\frac{2kT}{m}\right)^{\frac{3}{2}}\int d\mathbf{c}\, c_x c_z f(\mathbf{c},\mathbf{r}),$$

which in terms of g may be written

$$P_{xz} = \frac{2kT}{\sqrt{\pi}} n_0 \int_0^\infty dc\, c^4 e^{-c^2}\int_{-1}^{1} d\mu\, \mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x). \tag{16}$$

If now (12) is multiplied by $c^3\exp(-c^2)(1-\mu^2)^{\frac{1}{2}}$ and integrated over $c(0,\infty)$ and $\mu(-1,1)$, we find that

$$\int_0^\infty dc\, c^4 e^{-c^2}\int_{-1}^{1} d\mu\, \mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x) = \text{constant} \tag{17}$$

and hence that $P_{xz} =$ constant. The value of this constant will be obtained below.

The mean flow velocity $q(x)$ (i.e. the creep velocity) in the z direction is given by

$$q(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty dc\, c^3 e^{-c^2}\int_{-1}^{1} d\mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x), \tag{18}$$

where it should be remembered that q is in units of $(2kT_0/m)^{\frac{1}{2}}$.

The heat fluxes $Q_x(x)$ and $Q_z(x)$ in x and z directions, respectively, are given by

$$Q_x(x) = \frac{mn_0}{\sqrt{\pi}}\left(\frac{2kT_0}{m}\right)^{\frac{3}{2}}\int_0^\infty dc\, c^5 e^{-c^2}\int_{-1}^{1} d\mu\, \mu p(c,\mu,x) \tag{19}$$

and

$$Q_z(x) = \frac{mn_0}{2\sqrt{\pi}}\left(\frac{2kT_0}{m}\right)^{\frac{3}{2}}\int_0^\infty dc\, c^5 e^{-c^2}\int_{-1}^{1} d\mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x). \tag{20}$$

If now (11) is multiplied by $c^4 e^{-c^2}$ and integrated over $c(0,\infty)$ and $\mu(-1,1)$, we find that

$$\int_0^\infty dc\, c^5 e^{-c^2}\int_{-1}^{1} d\mu\, \mu p(c,\mu,x) = \text{constant} \tag{21}$$

and hence $Q_x(x) =$ constant, the value of which will be found below. It does not appear possible to obtain any simple relationship for $Q_z(x)$ which will have to be calculated in the same manner as $q(x)$.

Finally, the number density of the gas, $N(x,z)$, is given by

$$N(x,z) = n(x,z)\left[1+\frac{2}{\sqrt{\pi}}\int_0^\infty dc\, c^2 e^{-c^2}\int_{-1}^{1} d\mu\, p(c,\mu,x)\right] \tag{22}$$

and the temperature $\hat{T}(x,z)$ by

$$\hat{T}(x,z) = T(x,z)\left[1+\frac{4}{3\sqrt{\pi}}\int_0^\infty dc\, c^2(c^2-\tfrac{3}{2})\, e^{-c^2}\int_{-1}^{1} d\mu\, p(c,\mu,x)\right]. \tag{23}$$

It is clear from these definitions that (11) and (12) describe the phenomena of temperature jump and thermal creep respectively. Moreover, because of the simple boundary conditions employed there is no interaction between the two phenomena, which proceed independently. For a more general boundary condition this will not always be the case.

4. Thermal creep

4.1. *Asymptotic solution*

Let us consider equation (12) for thermal creep and note that the following expression is an acceptable solution:

$$g(c,\mu,x) = A_0 c(1-\mu^2)^{\frac{1}{2}} - K_z c a(c)(1-\mu^2)^{\frac{1}{2}} - \rho(c,\mu,x), \tag{24}$$

where A_0 is a constant and $a(c)$ satisfies the Chapman–Enskog thermal conductivity equation, i.e.

$$-c(c^2-\tfrac{5}{2}) + cV(c)\,a(c) = \int_0^\infty dc'\, c'^3 e^{-c'^2} K_1(c,c')\,a(c'), \tag{25}$$

where

$$\int_0^\infty dc\, c^4 e^{-c^2} a(c) = 0. \tag{26}$$

The function ρ is termed the transient solution and will tend to zero rapidly as x moves away from the boundary point. The particular form of the solution given by (24) can therefore be thought of as the sum of an asymptotic part (i.e. the first two terms of the right-hand side) and a transient. If (24) is inserted into (17), we find that

$$\int_0^\infty dc\, c^4 e^{-c^2} \int_{-1}^{1} d\mu\, \mu(1-\mu^2)^{\frac{1}{2}} \rho(c,\mu,x) = \text{constant} \tag{27}$$

and since this must be true for all x it is clear that the constant, and hence P_{xz}, are zero everywhere.

Similarly, inserting (24) into (18), we find

$$q(x) = \tfrac{1}{2}A_0 - \frac{1}{\sqrt{\pi}} \int_0^\infty dc\, c^3 e^{-c^2} \int_{-1}^{1} d\mu (1-\mu^2)^{\frac{1}{2}} \rho(c,\mu,x), \tag{28}$$

from which we may conclude that the asymptotic creep velocity q_{asy} is given by $\frac{1}{2}A_0$.

We shall consider the solution of (12) only for the case $\beta = 1$, i.e. purely diffuse reflexion. The extension to the more general case follows from the methods described in I and can be readily carried through. However, the case $\beta = 1$ has the advantage that an explicit expression for $q(x)$ may be obtained.

We can also write for the heat flux $Q_z(x)$:

$$Q_z(x) = \frac{2n_0 m}{3\sqrt{\pi}} \left(\frac{2kT_0}{m}\right)^{\frac{3}{2}} \left\{\frac{15\sqrt{\pi}}{16} A_0 - K_z \int_0^\infty dc\, c^6 a(c) e^{-c^2}\right\}$$
$$- \frac{mn_0}{2\sqrt{\pi}} \left(\frac{2kT_0}{m}\right)^{\frac{3}{2}} \int_0^\infty dc\, c^5 e^{-c^2} \int_{-1}^{1} d\mu (1-\mu^2)^{\frac{1}{2}} \rho(c,\mu,x). \tag{29}$$

For x sufficiently large, the contribution to Q_z from ρ becomes negligible and we can write

$$Q_z(\infty) = Q_z^{\text{asy}} = \tfrac{5}{2} p q_{\text{asy}} - \lambda_T\, dT/dz, \tag{30}$$

where now we have written q_{asy} in absolute units, and λ_T is the coefficient of thermal conductivity, defined by

$$\lambda_T = \frac{4kn_0}{3} \left(\frac{2kT_0}{\pi m}\right)^{\frac{1}{2}} \int_0^\infty dc\, c^6 e^{-c^2} a(c). \tag{31}$$

We see, therefore, that the rate of heat flow in the direction of decreasing temperature is reduced by the mass transport arising from thermal creep. We shall later calculate q_{asy} to find the magnitude of this effect.

4.2. *Solution of the equation for thermal creep*

If (24) is inserted into (12) and use made of (25), we find that $\rho(c,\mu,x)$ is given by

$$\left[c\mu\frac{\partial}{\partial x}+V(c)\right]\rho(c,\mu,x)$$
$$=\sum_{l=1}^{\infty}\frac{2l+1}{2l(l+1)}P_l^{(1)}(\mu)\int_0^{\infty}dc'\,c'^2e^{-c'^2}K_l(c,c')\int_{-1}^{1}d\mu'P_l^{(1)}(\mu')\rho(c',\mu',x) \quad (32)$$

subject to the boundary condition (for $\beta = 1$) of

$$\rho(c,\mu,0)=A_0c(1-\mu^2)^{\frac{1}{2}}-K_z ca(c)(1-\mu^2)^{\frac{1}{2}} \quad (\mu>0). \quad (33)$$

The procedure is now identical with that described in I. We use the Wiener–Hopf technique, with the generalized B.G.K. model for $K_1(c,c')$, and for the $\rho^*(c,\mu,0)$ of (37) of paper I we use the right-hand side of (33). Then, employing (A 2) of I, we obtain

$$A_0=\frac{3\gamma}{4}\Sigma_{\text{min}}\tau_-(0)K_z\int_0^{\infty}dc\,c^5e^{-c^2}\frac{a(c)}{\Sigma(c)}\int_0^1 d\mu\,\mu(1-\mu^2)\psi(c,\mu), \quad (34)$$

where the symbols in (34) are defined in I. The important point to note is that A_0 is given explicitly in terms of quadratures.

Knowledge of A_0 enables us to calculate all asymptotic properties of the gas exactly. It is also possible to obtain the detailed spatial variation of the flow and heat flux in the neighbourhood of the wall by inverting the appropriate Laplace transform. However, we shall not present the results here but simply note that $q(x)$ can be written in the form

$$q(x)=q_{\text{asy}}+\int_{\Sigma_{\text{min}}}^{\infty}I(t)\,e^{-tx}\,dt, \quad (35)$$

$I(t)$ being a rather complicated function. A similar expression is obtained for $Q_z(x)$. We can observe from this functional form that the asymptotic state is reached within about two maximum mean free paths from the boundary.

Fortunately, we can obtain a concise expression for the values of $q(0)$ and $Q_z(0)$ so that an adequate picture of the spatial variation of q and Q_z may be constructed.

Using (18) and (20) for $x = 0$, and the relation between the $\psi(c,\mu)$ functions, we find that

$$q(0)=\frac{3\gamma K_z}{4\sqrt{\pi}}\int_0^{\infty}dc'\,c'^5e^{-c'^2}a(c')\int_0^1\frac{d\mu'\mu'(1-\mu'^2)\psi(c',\mu')}{\mu\Sigma(c')+\mu'\Sigma(c)}$$
$$\times\int_0^{\infty}dc\,c^4e^{-c^2}\int_0^1 d\mu\,(1-\mu^2)\psi(c,\mu) \quad (36)$$

and
$$Q_z(0)=\frac{3mn_0\gamma K_z}{8\sqrt{\pi}}\left(\frac{2kT_0}{m}\right)^{\frac{3}{2}}\int_0^{\infty}dc\,c^6e^{-c^2}\int_0^1 d\mu\,(1-\mu^2)\psi(c,\mu)$$
$$\times\int_0^{\infty}dc'\,c'^5e^{-c'^2}\int_0^1\frac{d\mu'\,\mu'(1-\mu'^2)\psi(c',\mu')}{\mu\Sigma(c')+\mu'\Sigma(c)}(a(c')-a(c)). \quad (37)$$

4.3. *Application to limiting cases*

The two limiting cases that we shall consider are those of constant collision frequency, $V(c) = \lambda$, and constant cross-section, $V(c) = c\Sigma$. In both cases $a(c)$ is obtained from the B.G.K. model. First, therefore, we must solve (25) for $K_1(c, c') = \gamma cc' V(c) V(c')$. The equation for $a(c)$ is

$$-c(c^2-\tfrac{5}{2})+cV(c)\,a(c) = \gamma c V(c)\int_0^\infty dc'\,c'^4 e^{-c'^2} V(c')\,a(c') \tag{38}$$

and its general solution is

$$a(c) = a_0+(c^2-\tfrac{5}{2})/V(c), \tag{39}$$

where the arbitrary constant a_0 arises from the fact that the homogeneous equation has a non-trivial solution. a_0 is obtained from (26), whence

$$a(c) = -\frac{8}{3\sqrt{\pi}}\int_0^\infty dc'\,c'^4(c'^2-\tfrac{5}{2})\,e^{-c'^2}/V(c')+\left(\frac{c^2-\frac{5}{2}}{V(c)}\right). \tag{40}$$

For constant collision frequency

$$a(c) = (1/\lambda)\,(c^2-\tfrac{5}{2}), \tag{41}$$

whilst for constant cross-section

$$a(c) = \frac{2}{3\Sigma\sqrt{\pi}}+\left(\frac{c^2-\frac{5}{2}}{c\Sigma}\right). \tag{42}$$

We can evaluate the integrals in (34), (36) and (37), for constant collision frequency, by changing from polar co-ordinates in **c** back to Cartesian ones. Then we find that relation (A 1) of I becomes

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{dc_x\,c_x\,e^{-c_x^2}\,\tau_-(\lambda/c_x)}{\lambda+sc_x}\left(1+\frac{c_x}{\lambda}\Sigma_{\min}\right) = \frac{1}{(s+\Sigma_{\min})\,\tau_-(s)}. \tag{43}$$

Care must be exercised here in view of the fact that $\Sigma_{\min}$ for the constant collision frequency model is zero. We find that the limit must be taken after the various integrals have been evaluated.

In terms of the Cartesian co-ordinates, the expression for A_0 becomes

$$A_0 = \frac{K_z}{\lambda}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty dc_x\,c_x(c_x^2-\tfrac{1}{2})\,e^{-c_x^2}\tau_-(\lambda/c_x). \tag{44}$$

After repeated use of (43) we obtain finally

$$A_0 = \frac{K_z}{\lambda}\left[\frac{(\zeta\lambda)^2}{2}+\frac{1}{4}\right], \tag{45}$$

where $\zeta\lambda = \sigma$ is the slip coefficient defined in I.

The asymptotic velocity is therefore given by

$$q_{\text{asy}} = \frac{K_z}{2\lambda}\,[\tfrac{1}{2}\sigma^2+\tfrac{1}{4}]. \tag{46}$$

In absolute units we may write q_{asy} in terms of the B.G.K. viscosity μ_v as

$$q_{\text{asy}} = (\tfrac{1}{2}\sigma^2+\tfrac{1}{4})\frac{\mu_v R}{p}\frac{dT}{dz}, \tag{47}$$

where R is the gas constant and p the pressure. The factor in brackets is calculated as 0·7662 which is very close to the classical value of $\frac{3}{4}$ obtained by Kennard (1938) who used a less rigorous approach.

It may be more appropriate, in view of the fact that the creep velocity is induced by a temperature gradient, to write q_{asy} in terms of the B.G.K. conductivity. In that case we find

$$q_{\text{asy}} = (\tfrac{1}{2}\sigma^2 + \tfrac{1}{4})\frac{2}{5}\frac{\lambda_T}{p}\frac{dT}{dz}. \tag{48}$$

The surface velocity $q(0)$ is given in terms of Cartesian co-ordinates by

$$q(0) = \frac{3\gamma K_z}{16\sqrt{\pi}}\int_0^\infty dc_x\, e^{-c_x^2}\tau_-(\lambda/c_x)\int_0^\infty \frac{dc'_x\, c'_x(c'^2_x - \frac{1}{2})\, e^{-c'^2_x}}{c_x + c'_x}\tau_-(\lambda/c'_x) \tag{49}$$

and repeated use of (43) gives

$$q(0) = \frac{K_z}{2\lambda}\left(\frac{\sigma}{\sqrt{2}} - \frac{1}{2}\right), \tag{50}$$

whence

$$q(0) = \left(\frac{\sigma}{\sqrt{2}} - \frac{1}{2}\right)\frac{\mu_v R}{p}\frac{dT}{dz}, \tag{51}$$

or

$$q(0) = \left(\frac{\sigma}{\sqrt{2}} - \frac{1}{2}\right)\frac{2}{5}\frac{\lambda_T}{p}\frac{dT}{dz}. \tag{52}$$

We see therefore that the ratio $q_{\text{asy}}/q(0) \simeq 3{\cdot}5$ and hence there is a substantial velocity gradient in the neighbourhood of the surface.

The asymptotic heat flux can be obtained from (48) and (30) and is given by

$$Q_z^{\text{asy}} = -(\tfrac{3}{4} - \tfrac{1}{2}\sigma^2)\lambda_T\frac{dT}{dz}. \tag{53}$$

Thus there is still a net heat flow in the direction of decreasing temperature but mass flow has the effect of reducing the thermal conductivity by a factor of 0·234.

For $Q_z(0)$ we can reduce (37) to the following expression:

$$Q_z(0) = -f\lambda_T\frac{dT}{dz}, \tag{54}$$

where f is a constant given by

$$f = \tfrac{2}{5}[\tfrac{19}{8} - \sqrt{2}\,\sigma - \tfrac{1}{4}\sigma^2] = 0{\cdot}272. \tag{55}$$

Thus the heat flux at the surface is not too different from that of the asymptotic value.

Let us consider now the values of q and Q_z for the constant cross-section model. In this case $\Sigma_{\min} = \Sigma$ and (A 1) of I reduces to

$$\int_0^1 \frac{d\mu\,\mu(1-\mu^2)(1+\mu)\tau_-(\Sigma/\mu)}{\Sigma + s\mu} = \frac{4}{3}\frac{1}{(s+\Sigma)\tau_-(s)}. \tag{56}$$

In terms of $\tau_-(\Sigma/\mu)$ we then find that

$$A_0 = \frac{\sqrt{5}\,K_z}{2\Sigma\sqrt{\pi}}\int_0^1 d\mu\,\mu(1-\mu^2)(1+\mu)\tau_-(\Sigma/\mu), \tag{57}$$

whence, using (56), we obtain

$$A_0 = \frac{2K_z}{3\Sigma\sqrt{\pi}}. \tag{58}$$

The asymptotic flow velocity is therefore

$$q_{\text{asy}} = \frac{K_z}{3\Sigma\sqrt{\pi}} \tag{59}$$

or, in terms of the corresponding viscosity and thermal conductivity,

$$q_{\text{asy}} = \frac{5}{8}\frac{\mu_v R}{p}\frac{dT}{dz} \tag{60}$$

and

$$q_{\text{asy}} = \frac{2}{9}\frac{\lambda_T}{p}\frac{dT}{dz}. \tag{61}$$

The value of the velocity $q(0)$ at the surface is obtained after repeated use of (56) as

$$q(0) = \frac{K}{6\Sigma\sqrt{\pi}} \tag{62}$$

or

$$q(0) = \frac{5}{16}\frac{\mu_v R}{p}\frac{dT}{dz} \tag{63}$$

and

$$q(0) = \frac{1}{9}\frac{\lambda_T}{p}\frac{dT}{dz}. \tag{64}$$

We see therefore that the ratio $q_{\text{asy}}/q(0) = 2$. This ratio is evidently rather sensitive to the velocity dependence of the collision frequency.

The asymptotic heat flux is obtained easily as

$$Q_z^{\text{asy}} = -\frac{4}{9}\lambda_T\frac{dT}{dz} \tag{65}$$

and, as in the case of constant collision frequency, indicates that the thermal creep reduces the effective conduction; in this case by a factor of 0·444, which is about one half of the effect noted earlier.

For the surface value of $Q_z(0)$ we obtain from (37) and (56) the following expression:

$$Q_z(0) = -\frac{2}{9}\lambda_T\frac{dT}{dz}. \tag{66}$$

The conduction along the surface is therefore one half of its value in the main stream, which is in contrast to the result obtained from the constant collision frequency model.

At first sight, the fact that conduction is smaller in the region of lower velocity is puzzling. On the face of it, we might expect that the mass flow would always have a smaller effect there. However, we must recognize that the behaviour of the gas at the surface differs markedly from that in the free stream, in the sense that the former is in the Knudsen régime, and the latter in the hydrodynamic or Clausius limit. Thus in the Knudsen régime the conduction changes as well as the velocity and our results show that the extent to which this happens depends crucially on the velocity dependence of the mean free path.

5. Summary and conclusions

It has been shown that the conventional temperature jump problem takes on additional complications when the temperature gradient is inclined at an angle to the boundary wall. Whilst the normal component of the temperature gradient leads to the expected behaviour, as described by Welander and others, the component parallel to the wall induces a net mass flow into the gas in the direction of increasing temperature. This phenomenon has been described by Kennard as *thermal creep.*

The macroscopic properties of the thermal creep have been investigated and we have observed a large velocity gradient in the neighbourhood of the plate and, unlike slip flow or Poiseuille flow, the asymptotic velocity becomes spatially constant. Similar investigations have been made for the heat flux, which is found to be less than that expected in a stationary medium. It appears that the mass flow arising from thermal creep tends to reduce the heat conduction by a factor of between 2 and 4 in the asymptotic region and by a factor of about 4 at the boundary, depending on the velocity dependence of the collision frequency.

Finally, we add that, whilst the asymptotic part of the solution of the Boltzmann equation employed in this work is exact, the transient part is not. The equation for the transient part of the solution is approximated by the B.G.K. model which is deficient in certain features. We have, in a previous publication (Williams & Spain 1970), studied these deficiencies for the hard-sphere scattering model. Our results show that, for velocity perturbations (i.e. the creep-flow problem), the discrete eigenvalue spectra of hard-sphere and B.G.K. models are identical: namely, that there are no space eigenvalues, other than at zero, which correspond to the conservation laws. On the other hand, for temperature variations, the hard-sphere model discrete spectrum contains non-zero eigenvalues at $\pm 0{\cdot}975\Sigma_{\text{min}}$. This will cause an exponential term to appear in the solution which will decay less rapidly than the usual integral transient. The net effect is to increase slightly the thickness of the Knudsen layer for temperature problems. However, because the discrete eigenvalue is so close to the limit point, we do not expect this to have any serious consequences. The use of the velocity dependent B.G.K. model is therefore considered to be quite adequate.

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